The Plus Construction and Algebraic K-theory of Rings

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Abstract

The survey gives an introduction to algebraic K-theory. We first introduce the classical definition of K_0, K_1 groups and some properties of them. Then we apply Quillen's plus construction to construct general K_n and show that they are accord with the previous definitions when $n = 0, 1$. Finally, we give a brief introduction to K-theory spectrum.

Contents

1 Classical K-theory: K_0 and K_1

The beginning of algebraic K-theory is due to Grothendieck, in order to formulate his generalization of the Riemann-Roch Theorem to higher dimensions. He defined the Grothendieck group of a subcategory of an abelian category and in the case of rings, it becomes:

Definition 1.1 (K_0)**.** Let R be a ring. Let $F_0(R)$ be the free abelian group generated by $[P]$ of isomorphism classes of finitely generated projective modules over R. Let $K_0(R)$ be the quotient of $F_0(R)$ by the subgroup generated by all elements $[P] - [P'] - [P'']$ for every short exact sequence $0 \to P' \to P \to P'' \to 0$.

Actually Grothendieck defined K_0 first in the way of vector bundles. The two ways of definitions are the same due to an important theorem in topological K-theory relating vector bundles and projective modules:

Theorem 1.2 (Swan)**.** *Let X be a compact Hausdorff space. The category of real vector bundle over X is equivalent to the category of finitely generated projective module over $C^\infty(X)$. The correspondence is given by sending a vector bundle to the $C^\infty(X)$-module of smooth sections.*

Under such identification, K_0 can also be considered as a group of vector bundles over a compact Hausdorff space X. Atiyah and Hirzebruch, motivated by Grothendieck's work, defined higher topological K-groups $K^{-n} := K_0(S^n X)$, where SX is the suspension of X. Later, Bass gave the definition of $K_1(R)$ as an algebraic analogue of topological K^{-1} by the following observation:

Observation. *A vector bundle on SX is trivial on each cone CX since CX is contractible. Thus, the structure of the vector bundle is determined by how bundles are glued on the middle X.*

Converting this into language of projective modules, we get:

Definition 1.3 (K_1)**.** Let R be a ring. Let $F_1(R)$ be the free abelian group generated by isomorphism classes of maps $[P \xrightarrow{\alpha} P]$ where P is a finitely generated projective module and α is an isomorphism. Let $K_1(R)$ be the quotient of $F_1(R)$ by the following relations:

(a) $[P \xrightarrow{\alpha} P] = [P' \xrightarrow{\alpha'} P'] + [P'' \xrightarrow{\alpha''} P'']$ whenever there is a short exact sequence $0 \to P' \to P \to P'' \to 0$.

(b) $[P \xrightarrow{\alpha\beta} P] = [P \xrightarrow{\alpha} P] + [P \xrightarrow{\beta} P]$.

There is another formulation of $K_1(R)$ using infinite general linear group:

Observation. *There is a map $GL_n(R) \to K_1(R)$ given by $A \mapsto [R^n \xrightarrow{A} R^n]$. Relation (b) guarantees that this is a group homomorphism.*

Definition 1.4 (Infinite general linear group)**.** Let R be a ring. Let $j_n \colon GL_n(R) \to GL_{n+1}(R)$ be the inclusion that adds an additional row and column and a 1 along the diagonal. The infinite general linear group $GL(R)$ is the colimit:

$$GL(R) := \text{colim}[GL_1(R) \xrightarrow{j_1} GL_2(R) \xrightarrow{j_2} GL_3(R) \xrightarrow{j_3} \cdots]$$

Lemma 1.5 (Whitehead)**.** *Let $E(R)$ be the subgroup of $GL(R)$ generated by the elementary matrices. Then $E(R) = [GL(R), GL(R)]$.*

Proof. See [3, Lemma 7.10]. □

Theorem 1.6. *The natural map*

$$GL(R)^{ab} = GL(R)/E(R) \to K_1(R)$$

is an isomorphism.

Proof. See [3, Theorem 7.6]. □

Definition 1.7 (Relative K-group)**.** Let R be a ring and $I \subset R$ a two-sided ideal. The double of R along I is the subring of the Cartesian product $R \times R$ given by

$$D(R, I) := \{(x, y) \in R \times R : x - y \in I\}$$

Let $p_1 \colon D(R, I) \to R$ be the projection to the first component. For $i = 0, 1$, define the relative K_i-group of R along I to be

$$K_i(R, I) := \ker\Big((p_1)_* \colon K_i\big(D(R, I)\big) \to K_i(R)\Big)$$

Theorem 1.8. *Let R be a ring and $I \subset R$ a two-sided ideal. Then there is a exact sequence:*

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow K_0(R, I) \longrightarrow K_0(R) \longrightarrow K_0(R/I)$$

Proof. See [7, Theorem 2.5.4]. □

This reminds us long exact sequences of homology and homotopy groups. It natural to ask: Can we extend this sequence further? What are the higher K-groups?

2 Quillen's Plus Construction

Quillen's plus-construction arises from his proof of the Adams Conjecture, using ideas from Sullivan. This gives us a way to construct higher K-groups.

Definition 2.1 (Perfect group)**.** A group is called perfect if it is equal to its commutator subgroup.

A perfect radical of a group G is the largest perfect subgroup, denoted by $P[G]$.

Theorem 2.2 (Quillen's plus construction)**.** *Let X be a connected CW complex with base point x_0. Let N be a perfect normal subgroup of $\pi_1 := \pi_1(X, x_0)$. Then there is a CW complex X_N^+ by attaching 2-cells and 3-cells to X with inclusion $i\colon X \to X_N^+$, such that*

(a) The map $i_*\colon \pi_1(X, x_0) \to \pi_1(X_N^+, x_0)$ is the quotient map $\pi_1 \to \pi_1/N$.

(b) For any local coefficients L, i.e., L is an $\mathbb{Z}[\pi/N]$-module, the maps $i_*\colon H_n\big(X; i^*(L)\big) \to H_n(X_N^+; L)$ are isomorphisms for all n.

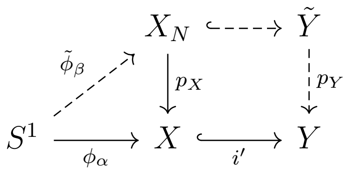
Furthermore, i is initial in the homotopy category among all maps $f\colon (X, x_0) \to (Y, y_0)$, where Y is connected and $f_*(N) = \{1\}$ in $\pi_1(Y, y_0)$. In particular, X_N^+ is unique up to homotopy.

If $N = P[\pi_1]$, we denote it by X^+.

Proof. We want to attatch 2-cells to X to kill N in π_1. Then we attatch 3-cells to remedy these in cellular homology.

First suppose $\phi_\alpha\colon S^1 \to X$ are generators of N. We use ϕ_α as attatching maps of 2-cells to X to form a new CW complex Y, so $\pi_1(Y, x_0) = \pi_1/N$.

Next, we make use of a covering space of X to construct X^+. By [6, Chapter 3.8], there is a covering map $p_N\colon X_N \to X$ such that $\pi_1(X_N, x_N) = N$. For each cycle $\tilde{\psi}_\beta\colon S^1 \to X_N$ that projects to some ϕ_α, we attatch a 2-cell $\tilde{e}_\beta^2$ using $\tilde{\psi}_\beta$ as attaching map to get a simply-connected space $\tilde{Y}$. Since $i \circ \phi_\alpha$ induces zero map on fundamental groups and $\tilde{Y}$ is built from X_N by attaching 2-cells, $i \circ p_X$ extends to $\tilde{Y}$. Denote the extended map $p_Y\colon \tilde{Y} \to Y$. By construction, p_Y is a covering map, so $\tilde{Y}$ is actually the universal covering of Y. Then we have the following commutative diagram:



Note that $H_1(X_N; \mathbb{Z}) = \pi_1(X_N, x_N)^{ab} = 0$. There is a diagram:

$$\begin{array}{ccccc} \pi_1(\tilde{Y}, \tilde{y}) & & & & \\ \big\downarrow h & & & & \\ H_2(\tilde{Y}; \mathbb{Z}) & \xrightarrow{j} & H_2(\tilde{Y}, X_N; \mathbb{Z}) & \xrightarrow{\partial} & H_1(X_N; \mathbb{Z}) = 0 \end{array}$$

where h is the Hurewicz map. Since $\tilde{Y}$ is simply-connected, h is an isomorphism by Hurewicz Theorem. Thus, $j \circ h$ is a surjection.

Note that by cellular homology, $H_2(\tilde{Y}, X_N; \mathbb{Z})$ is a free abelian group with basis $[\tilde{e}^2_\beta]$. Pick $\tilde{\psi}_\beta \colon S^2 \to \tilde{Y}$ such that $(j \circ h)_*(\tilde{\psi}_\beta) = [\tilde{e}^2_\beta]$. Let $\psi_\alpha = p_Y \circ \tilde{\psi}_\alpha$. Finally by attatching 3-cells e^3_β onto Y using ψ_β as attaching maps, we get the space X^+. It remains to show that X^+ satisfies conditions (a)(b).

(a) Since X^+ is built from Y by attatching 3-cells, $\pi_1(X^+, x_0) = \pi_1(Y, x_0) = \pi_1/N$.

(b) Construct the universal covering $\widetilde{X^+}$ of X^+ by attaching 3-cells $\tilde{e}^3_\beta$ onto $\tilde{Y}$ as we constructed $\tilde{Y}$. We get the following commutative diagram:

$$\begin{array}{ccccc} X_N & \hookrightarrow & \tilde{Y} & \hookrightarrow & \widetilde{X^+} \\ \big\downarrow p_X & & \big\downarrow p_Y & & \big\downarrow p_{X^+} \\ X & \underset{i'}{\hookrightarrow} & Y & \underset{i''}{\hookrightarrow} & X^+ \end{array}$$

We first show that $H_*(\widetilde{X^+}, X_N; \mathbb{Z}) = 0$. Since $\widetilde{X^+}$ is built from X_N by attaching only 2 and 3-cells, the relative cellular complex of the pair looks like:

$$\cdots \longrightarrow 0 \longrightarrow C_3(\widetilde{X^+}, X_N) \xrightarrow{d} C_2(\widetilde{X^+}, X_N) \longrightarrow 0 \longrightarrow 0$$

It remains to show that d is an isomorphism. Since $\tilde{Y}$ differs from X_N only by 2-cells and from $\widetilde{X^+}$ only by 3-cells, we have

$$C_3(\widetilde{X^+}, X_N) = C_3(\widetilde{X^+}, \tilde{Y}) = H_3(\widetilde{X^+}, \tilde{Y}; \mathbb{Z})$$
$$C_2(\widetilde{X^+}, X_N) = C_2(\tilde{Y}, X_N) = H_2(\tilde{Y}, X_N; \mathbb{Z})$$

Then the differential d is the same with the composition:

$$H_3(\widetilde{X^+}, \tilde{Y}; \mathbb{Z}) \xrightarrow{\partial} H_2(\tilde{Y}; \mathbb{Z}) \xrightarrow{j} H_2(\tilde{Y}, X_N; \mathbb{Z})$$

Note that $H_3(\widetilde{X^+}, \tilde{Y}; \mathbb{Z})$ is a free abelian group generated by $[\tilde{e}^3_\beta]$. Since we attatched the $\tilde{e}^3_\beta$ onto $\tilde{Y}$ by $\tilde{\psi}_\beta$, we have $\partial([\tilde{e}^3_\beta]) = h(\tilde{\psi}_\beta)$. Thus,

$$d([\tilde{e}^3_\beta]) = (j \circ \partial)([\tilde{e}^3_\beta]) = (j \circ h)(\tilde{\psi}_\beta) = [\tilde{e}^2_\beta]$$

This shows that d is a isomorphism.

Since, $H_n(X^+, X; L)$ is the homology of $C_n(\widetilde{X^+}, X) \otimes_{\mathbb{Z}[\pi/N]} L$, so $H_n(X^+, X; L) = 0$ for all n by [2, Theorem 5.13]

For the universal property, the idea is that $H^n\big(X^+_N, X; \pi_n(Y)\big) = 0$ for all n and try to apply the obstruction theory. However, here Y may not be an abelian space, which requires a general form of obstruction theory and we do not want to prove it here. Thus, we refer to [5, Proposition 1.1.2] □

Proposition 2.3. *The plus construction is functorial, i.e., if $f\colon (X, x_0) \to (Y, y_0)$ is a based map of connected CW complexes ad $f_*(N) \subset N'$, where $N \subset \pi_1(X, x_0), N' \subset \pi_1(Y, y_0)$ are perfect normal subgroups, then there exists a map $f^+\colon X^+_N \to Y^+_{N'}$, unique up to homotopy such that $i_Y \circ f = f^+ \circ i_X$, where $i_X\colon X \to X^+_N, i_Y\colon Y \to Y^+_{N'}$ are inclusions.*

Proof. Apply the universal property to the composite map $i_Y \circ f$. □

Definition 2.4. (Higher algberaic K-groups) Let

$$\mathbf{K}(R) := K_0(R) \times BGL(R)^+ \text{ and } K_n(R) := \pi_n\big(\mathbf{K}(R)\big)$$

for $n \in \mathbb{Z}_{\geqslant 0}$, where $BGL(R)$ is the classifying space of $GL(R)$.

We need to verify that such definition accords with the previous definitions in the first section. This is obvious for K_0. For K_1, it is easy to see that $E(R)$ is a perfect group, so $E(R)$ is the perfect radical of $GL(R)$. Thus,

$$\pi_1\big(\mathbf{K}(R)\big) = \pi_1\big(BGL(R)^+\big) = GL(R)/P[GL(R)] = GL(R)/E(R) = K_1(R)$$

Definition 2.5 (Higher relative K-group)**.** For a ring R and a two-sided ideal I of R, let $\pi\colon R \to R/I$ be the natural projection. Then π induces $\pi^+\colon \mathbf{K}(R) \to \pi^+\big(\mathbf{K}(R)\big) \subset \mathbf{K}(R/I)$ by Proposition 2.3. Let $\mathbf{K}(R, I)$ be the homotopy fiber of π^+ and define $K_n(R, I) := \pi_n\big(\mathbf{K}(R, I)\big)$.

This gives the long exact sequence:

$$\cdots \to K_{n+1}(R/I) \to K_n(R, I) \to K_n(R) \to K_n(R/I) \to \cdots$$

3 An Introduction to K-theory Spectrum

The last section is a reformulation of higher K-groups coherent with the definition by Atiyah and Hirzebruch in the first section using the suspension of a space. We want to define a suspension of a ring R. First we define the suspension of $\mathbb{Z}$:

Definition 3.1 (Suspension of $\mathbb{Z}$)**.** The cone $C\mathbb{Z}$ of $\mathbb{Z}$ is the set of all infinite matrices with integral coefficients having only a finite number of non-trivial elements on each row and on each column, which admits a ring structure under the usual addition and multiplication of matrices. Let $J\mathbb{Z}$ be the ideal of $C\mathbb{Z}$ which consists of all matrics having only finitely many non-trivial coefficients. Finally, let us define the suspension of $\mathbb{Z}$ to be the quotient ring $\Sigma\mathbb{Z} := C\mathbb{Z}/J\mathbb{Z}$.

Definition 3.2 (Suspension of R)**.** Let R be a ring. The suspension of R is the ring $\Sigma R := \Sigma\mathbb{Z} \otimes_{\mathbb{Z}} R$.

Proposition 3.3. *Let R be a ring. There is a natural homotopy equivalence:*

$$\mathbf{K}(R) \cong \Omega\big(\mathbf{K}(\Sigma R)\big)$$

Thus, $\mathbf{K}(R)$ is an infintie loop space.

Proof. See [1, Theorem 4.9]. □

Corollary 3.4. *For any ring R and any integer $i \geqslant 0$, there is an isomorphism*

$$K_i(R) \cong K_{i+1}(\Sigma R)$$

Proof. By definition,

$$K_i(R) = \pi_i\big(\mathbf{K}(R)\big) = \pi_i\Big(\Omega\big(\mathbf{K}(\Sigma R)\big)\Big) = \pi_{i+1}\big(\mathbf{K}(\Sigma R)\big) = K_{i+1}(\Sigma R)$$

□

Definition 3.5 (K-theory spectrum)**.** Let R be a ring. The K-theory spectrum of R is the spectrum K_R whose n-th space is $(K_R)_n = \mathbf{K}(\Sigma R)$ for all $n \geqslant 0$.

References

[1] Dominique Arlettaz. Algebraic k-theory of rings from a topological viewpoint. *Publicacions matemàtiques*, 44(1):3–84, 2000. 3.3

[2] James F. Davis. *Lecture notes in algebraic topology*. Graduate studies in mathematics, v. 35. American Mathematical Society, 2001. 2.2

[3] Daniel Dugger. A geometric introduction to k-theory. https://pages.uoregon.edu/ddugger/kgeom.pdf. 1.5, 1.6

[4] Daniel R Grayson. Quillen's work in algebraic k-theory. *Journal of K-Theory*, 11(3):527–547, 2013.

[5] Jean-Louis Loday. K-théorie algébrique et représentations de groupes. *Annales scientifiques de l'École normale supérieure*, 9(3):309–377, 1976. 2.2

[6] J. Peter. May. *A concise course in algebraic topology*. Chicago lectures in mathematics. University of Chicago Press, 1999. 2.2

[7] Jonathan M. Rosenberg. *Algebraic K-theory and its applications*. Graduate texts in mathematics ; 147. Springer-Verlag, 1994. 1.8

[8] Shan Shah. The quillen plus-construction in algebraic k-theory. https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.1069.8799&rep=rep1&type=pdf, 2012.